2023-24 MATH2048: Honours Linear Algebra II

Homework 3

Due: 2023-09-29 (Friday) 23:59

For the following homework questions, please give reasons in your solutions. Scan your solutions and submit it via the Blackboard system before due date. All questions are selected from Friedberg §2.1-2.2.

1. Prove that there exists a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that $T(1,1) = (1,0,2)$ and $T(2,3) = (1,-1,4)$. What is $T(8,11)$?

2. Let V be a finite-dimensional vector space and $T : V \to V$ be linear.

   (a) Suppose that $V = R(T) + N(T)$. Prove that $V = R(T) \oplus N(T)$.

   (b) Suppose that $R(T) \cap N(T) = \{0\}$. Prove that $V = R(T) \oplus N(T)$.

   (c) Give an example of V and T such that $V = R(T) \oplus N(T)$.

   Be careful to say in part (a)(b) where finite-dimensionality is used.

3. Let V be an n-dimensional vector space with an ordered basis β. Define $T : V \to F^n$ by $T(x) = [x]_\beta$. Prove that T is linear.

4. Let V be a vector space with the ordered basis $\beta = \{v_1, v_2, ..., v_n\}$. Define $v_0 = 0$. By Theorem 2.6 (p. 72), there exists a linear transformation $T : V \to V$ such that $T(v_j) = v_j + v_{j-1}$ for $j = 1, 2, ..., n$. Compute $[T]_\beta$.

5. Let V and W be vector spaces such that $\dim(V) = \dim(W)$, and let $T : V \to W$ be linear. Show that there exist ordered bases β and γ for V and W, respectively, such that $[T]_\beta^\gamma$ is a diagonal matrix.

The following are extra recommended exercises not included in homework.. All questions except Q4 are selected from Friedberg §2.1-2.3.

1. Let V be a finite-dimensional vector space, and let $T : V \to V$ be linear.

(a) If $\text{rank}(T) = \text{rank}(T^2)$, prove that $R(T) \cap N(T) = \{0\}$. Deduce that $V = R(T) \oplus N(T)$.

(b) Prove that $V = R(T^k) \oplus N(T^k)$ for some positive integer k.

2. Let A and B be $n \times n$ matrices. Recall that the trace of A is defined by $\text{tr}(A) = \sum_{i=1}^{n} A_{ii}$. Prove that $\text{tr}(AB) = \text{tr}(BA)$ and $\text{tr}(A) = \text{tr}(A^t)$.

3. Let V be an n-dimensional vector space, and let $T : V \to V$ be a linear transformation. Suppose that W is a T-invariant subspace of V having dimension k. Show that there is a basis β for V such that $[T]_\beta$ has the form

$$\begin{pmatrix} A & B \\ O & C \end{pmatrix},$$

where A is a $k \times k$ matrix and O is the $(n-k) \times k$ zero matrix.

4. If V and W are vector spaces over the same field F, prove that $V \times W$ is isomorphic to $W \times V$,i.e, there is a bijective linear map between $V \times W$ and $W \times V$.

5. Label the following statements as true or false. Assume that V and W are finite-dimensional vector spaces with ordered bases β and γ, respectively, and $T, U : V \to W$ are linear transformations.

(a) For any scalar a, $aT + U$ is a linear transformation from V to W.

(b) $[T]_\beta^\gamma = [U]_\beta^\gamma$ implies that $T = U$.

(c) If $m = \dim(V)$ and $n = \dim(W)$, then $[T]_\beta^\gamma$ is an $m \times n$ matrix.

(d) $[T + U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$.

(e) $L(V, W)$ is a vector space.

(f) $L(V, W) = L(W, V)$.